MULTIMEDIA UNIVERSITY

STUDENT ID NO

MULTIMEDIA UNIVERSITY

FINAL EXAMINATION

TRIMESTER 2, 2017/2018

EME3156-COMPUTATIONAL METHODS FOR MECHANICAL ENGINEERING
(ME)

12 MARCH 2018
2.30 p.m. – 4.30 p.m.
(2 Hours)

INSTRUCTION TO STUDENT

1. This question paper consists of 10 pages excluding the cover page.
2. This paper is divided into **Sections A and B**. Answer all questions in Section A. Choose two questions to answer in Section B. The distribution of the marks for each question is given.
3. Please write all your answers in the answer booklet provided.
4. All necessary workings **MUST** be shown.

Section A

Question 1

Equation (1.1) is a first order ordinary differential equation (ODE) with initial condition,

$$\frac{dy}{dx} - 1 - \frac{y}{x} = 0 \ , \ y(1) = 2 \tag{1.1}$$

a) Find $y(1.4)$ using predictor-corrector method, with step size $h = 0.2$.

[8 marks]

b) Find $y(1.4)$ using fourth order Runge-Kutta, RK4, with step size $h = 0.4$.

[8 marks]

c) Compare your result above, a) and b), with exact solution in terms of accuracy. Exact solution is given by $y(x) = x \ln x + 2x$.

[4 marks]

d) Find the first and second derivatives of the function:

$$f(x,y) = 1 + \frac{y}{x} \tag{1.2}$$

Thus, by using Taylor expansion of order 2, write down the expression for the solution y. The first and second derivatives of the function $f(x,y)$ must be included in the said solution.

[5 marks]

Continued...

Question 2

a) Equation (2.1) shows a second order ODE to represent one-dimensional heat transfer problem with the boundary conditions where dependent variable T is the temperature. In order to solve the problem, shooting method is used.

$$\frac{d^2T}{dx^2} - 3T = 0,\ T(0) = 100,\ T(1) = 0 \tag{2.1}$$

i. Second order ODE is converted into two first order ODE with one assumption of initial condition. Write down two first order ODE equations together with the initial conditions. The value of the assumed initial condition is set to zero. Two new variables "U" and "V" are needed.

[2 marks]

ii. Using the explicit Euler method with the step size $h = 0.25$, find $T(1)$.

[8 marks]

iii. It is unlikely to obtain $T(1)$ from (ii) that matches with the given value in Equation (2.1). Thus, without doing further calculations, explain the next steps so that $T(1) = 0$ is obtained.

[2 marks]

Continued...

b) Newton method and Secant method can be used to solve a non-linear system of equations. In Newton's method, given an initial approximation, $x^{(0)}$, for a root of that function, it computes a sequence of improved approximations to the root. Since the non-linear system of equations appear is a set of simultaneous equations, these equations are then assembled to become a matrix equation $[J]\{\Delta x\} = \{F\}$

A nonlinear system of equations is given:

$$\begin{aligned} x_1^3 - 2x_2 - 2 &= 0 \\ x_1^3 - 5x_3^2 + 7 &= 0 \\ x_2 x_3^2 - 1 &= 0 \end{aligned} \tag{2.2}$$

i. Write down the Newton method for solving nonlinear system.

[3 marks]

ii. Find the Jacobian matrix for the system, $\left[J\left(x^{(0)}\right)\right]$ when the initial guess is given as $x^{(0)} = [1 \;\; 1 \;\; 1]^T$. Determine also $F\left(x^{(0)}\right)$.

[5 marks]

iii. Determine the improved approximation $x^{(1)}$.

[5 marks]

Continued...

Section B

Question 3

Equation (3.1) below is a one-dimensional heat equation. It is used to model temperature distribution measured along a homogeneous rod. The constant α is the ratio of heat conduction to the heat storage. The independent variables are x and t.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad , 0 \le x \le 1 \tag{3.1}$$

a) Convert the equation above into **implicit** finite difference form using central difference scheme in space and forward difference scheme in time.

[4 marks]

b) State the stability criteria for **implicit** finite difference method. **[2 marks]**

A problem is to be modeled using heat equation above. α is taken as 1 while the spatial domain is to be discretized into six equally spaced nodes. Use time interval of $\Delta t = 0.02$ for calculation. The initial and boundary conditions are given by

$$u(x,0) = 5\sin x, \quad 0 \le x \le 1$$
$$u(0,t) = 0, \quad 0 \le t \le 1$$
$$u(1.0,t) = (\sin(1))e^{-t}, 0 \le t \le 1$$

c) Present the finite difference form (from part (a)) in matrix form, as given by the equation (3.2) below..

$$\{u\}^t = [A]\{u\}^{t+1} + \{\text{Boundary Conditions}\} \tag{3.2}$$

$\{u\}^{t+1}$ and $\{u\}^t$ are vectors for nodal variables at time t+1 and t respectively. $[A]$ is a square matrix. **[4 marks]**

Continued...

d) Solve for vector for nodal variables $\{u\}^1$ and $\{u\}^2$ at time t+1 and t +2, given that the inverse matrix for part (c) is given by

$$\frac{1}{209}\begin{bmatrix} 112 & 30 & 8 & 2 \\ 30 & 120 & 32 & 8 \\ 8 & 32 & 120 & 30 \\ 2 & 8 & 30 & 112 \end{bmatrix} \text{ or } \begin{bmatrix} 0.536 & 0.144 & 0.038 & 0.01 \\ 0.144 & 0.574 & 0.153 & 0.038 \\ 0.038 & 0.153 & 0.574 & 0.144 \\ 0.01 & 0.038 & 0.144 & 0.536 \end{bmatrix}$$

[15 marks]

Continued...

Question 4:

A wave traveling in forward direction in a domain is given by

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad t \ge 0 \tag{4.1}$$

The velocity, boundary and initial conditions are

$$u(0,t) = u(1,t) = 0, \; u(x,0) = \sin(\pi x), \; \frac{\partial u}{\partial t}(x,0) = 3\pi \sin \pi x$$

The partial differential equation above is to be solved using finite difference method. The spatial domain is to be discretized into 6 nodes equally spaced and to calculate the wave progression in time, use $\Delta t = 0.02$.

a) By using central difference of the second derivative, approximate the derivative $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$ in their respective variables. The subscripts should represent spatial location (x) while the superscripts should represent time steps (t).

[7 marks]

b) Incorporating initial and boundary conditions into the problem, rewrite equation (4.1) in matrix form, given as:

$$\{u\}^{t+1} = [A]\{u\}^{t} + \{u\}^{t-1} + \{\text{boundary conditions}\}$$

where $\{u\}^{t+1}$, $\{u\}^{t}$ and $\{u\}^{t-1}$ are vectors for nodal variables at time t+1, t and t-1 respectively. $[A]$ is a square matrix. **[13 marks]**

c) Evaluate $\{u\}^{1}$, vector for nodal variables at time t +1.

[5 marks]

Continued...

Question 5

Temperature distribution of a two-dimensional irregular metal can be expressed in terms of a steady state diffusion equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{f(x,y)}{a}=0 \qquad (5.1)$$

where $f(x,y)$ is the power density at (x,y) and a is the thermal conductivity. For this problem, $\frac{f(x,y)}{a}=0$

a) Derive, by using central difference of the second derivative, the finite difference form of the equation (5.1) **[5 marks]**

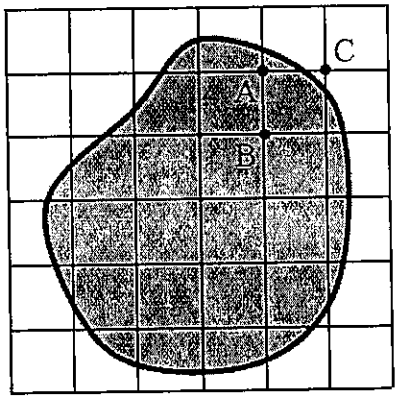

Figure 5.1

b) Figure 5.1 shows an irregular-shaped domain (in dark shade) drawn on grid lines. Nodes have to be defined in the domain before it can be solved numerically. If a point (the intersection of two grid lines) falls in the domain, it counts as a node. Example: point A and B are counted as nodes, but point C is not. Furthermore, point A is considered as a boundary node.

Sketch the Figure 5.1 in your answer booklet and complete the node labelling by making a circle ( ) as shown above. The nodes must be numbered in sequence starting from "1" on the bottom left. The irregular shape can be excluded from your sketch. **[5 marks]**

Continued...

c) State all the boundary nodes based on the node numbering in part (b).

[4 marks]

d) The finite difference equation used to model the discretization is what you derived in part (a). Write down the finite difference equations of the domain, and later rewrite the equation in matrix form, given by,

$$[A]\{u\} = \{c\} \tag{3.2}$$

where $\{u\}$ is a vector for nodal variable at a two-dimensional location and $\{c\}$ is the vector after including boundaries. $[A]$ is a square matrix. DO NOT ATTEMPT TO SOLVE IT but only reproduce the Laplace equation (3.1) in matrix form.

[11 marks]

Continued...

Appendix

Explicit Euler method

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

Implicit Euler method

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

Modified Euler method (predictor-corrector)

Step 1: $K_1 = f(x_i, y_i),\ y_{i+1}^P = y_i + hK_1$

Step 2: $K_2 = f(x_{i+1}, y_{i+1}^P),\ y_{i+1}^C = y_i + h\frac{1}{2}(K_1 + K_2)$

Midpoint method

Step 1: $K_1 = f(x_i, y_i),\ y_m = y_i + \frac{h}{2}K_1$

Step 2: $K_2 = f(x_m, y_m),\ y_{i+1} = y_i + hK_2$

Fourth-order Runge-Kutta

Step 1: $K_1 = f(x_i, y_i),\ y_{m1} = y_i + \frac{h}{2}K_1$

Step 2: $K_2 = f(x_m, y_{m1}),\ y_{m2} = y_i + \frac{h}{2}K_2$

Step 3: $K_3 = f(x_m, y_{m2}),\ y_{m3} = y_i + hK_3$

Step 4: $K_4 = f(x_{i+1}, y_{m3}),\ y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Second-order Adams-Bashforth

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

Third-order Adams-Bashforth

$$y_{i+1} = y_i + \frac{h}{12}(23f_i - 16f_{i-1} + 5f_{i-2})$$

Fourth-order Adams-Bashforth

$$y_{i+1} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

Second-order Adams-Moulton

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1})$$

Third-order Adams-Moulton

$$y_{i+1} = y_i + \frac{h}{12}(5f_{i+1} + 8f_i - f_{i-1})$$

Fourth-order Adams-Moulton

$$y_{i+1} = y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2})$$

Continued...

First derivative, forward difference

$$y' = \frac{1}{h}\left(y_{i+1} - y_i\right)$$

First derivative, backward difference

$$y' = \frac{1}{h}\left(y_i - y_{i-1}\right)$$

First derivative, central difference

$$y' = \frac{1}{2h}\left(y_{i+1} - y_{i-1}\right)$$

Second derivative, forward difference

$$y'' = \frac{1}{h^2}\left(y_i - 2y_{i+1} + y_{i+2}\right)$$

Second derivative, backward difference

$$y'' = \frac{1}{h^2}\left(y_{i-2} - 2y_{i-1} + y_i\right)$$

Second derivative, central difference

$$y'' = \frac{1}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right)$$

Jacobi method

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=1+1}^{n} a_{ij}x_j^k\right)$$

Gauss-Seidel method

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=1+1}^{n} a_{ij}x_j^k\right)$$

Successive over-relaxation method

$$x_i^{k+1} = (1-\omega)x_i^k + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{k+1} - \sum_{j=1+1}^{n} a_{ij}x_j^k\right)$$

Jacobian matrix

$$[J] = \begin{bmatrix} \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} & \Lambda & \frac{\partial f_i}{\partial x_n} \\ \frac{\partial g_i}{\partial x_1} & \frac{\partial g_i}{\partial x_2} & \Lambda & \frac{\partial g_i}{\partial x_n} \\ \frac{\partial h_i}{\partial x_1} & \frac{\partial h_i}{\partial x_2} & \Lambda & \frac{\partial h_i}{\partial x_n} \\ \mathrm{M} & \mathrm{M} & \Lambda & \mathrm{M} \end{bmatrix}$$

Sherman-Morrison formula

$$[J]_{\text{new}}^{-1} = [J]_{\text{old}}^{-1} + \frac{\left\{\{\Delta x\} - [J]_{\text{old}}^{-1}\{\Delta F\}\right\}\{\Delta x\}^{\mathrm{T}}[J]_{\text{old}}^{-1}}{\{\Delta x\}^{\mathrm{T}}[J]_{\text{old}}^{-1}\{\Delta F\}}$$

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